This article can be cited as K. Challita, Infinite RCC8 Networks, International Journal of Artificial Intelligence, vol. 15, no. 1, pp. 147-162, 2017.

Infinite RCC8 Networks

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Abstract

In this paper, we solve the problem of determining a consistent instantiation of any path-consistent and atomic network of $RCC8$ base relations. We already know that such networks that contain a finite number of variables have a realization in any dimension $d \geq 1$. The novelty of our work is that given any path-consistent and atomic constraint $RCC8$ network, possibly with countably infinite variables, we are able to construct in polynomial time a realization that satisfies it. For this purpose, we suitably instantiate the variables of such a network by associating to them some regular closed sets of the set of real numbers.

Keywords: Region Connection Calculus, Constraint satisfaction problems, Qualitative spatial reasoning, Iterative Deepening Search.

1998 Computing Subject Classification: F1, F2, F4.

1 Introduction

Allen's (Allen, 1983) seminal work on qualitative temporal reasoning paved the way to other researchers in Artificial Intelligence to develop a wide variety of temporal and spatial formalisms that allow us to reason about objects with respect to time and space (Ligozat, 1991; Vilain and Kautz, 1986; Skiadopoulos and Koubarakis, 2004). Indeed, for some specific systems, it appears that only the qualitative representation of their temporal information suffices to describe them. This is also true for space. The increasing interest concerning qualitative spatial reasoning is due to its multitude of applications to different fields such as Geographic Information Systems, robot navigation, high level vision or natural languages. Several formalisms (Asher and Vieu, 1995; Balbiani, Condotta and del Cerro, 1999; Condotta, 2000; Ligozat and Renz, 2004; Farhat and Feuillade, 2015) allow us to qualitatively describe objects in space and to reason about their respective positions. The works of Clarke (Clarke, 1981; Clarke, 1985) have been followed by those of Randell, Cui and Cohn (Randell, Cui and Cohn, 1992), who developed the RCC formalism. This formalism studies the different relations that we can define between regions in a topological space, based on the primitive relation of connection. Two of its fragments, namely $RCC5$ and $RCC8$, were introduced later on by Bennett (Bennett, 1994). Since then, several real-life applications of these two formalisms have been found by other researchers. For example, Bruno Bouzy (Bouzy, 2001) used $RCC8$ in programming the Go game, and Andreas et al. (Lattner, Timm, Lorenz and Herzog, 2005) used R$CC5$ in order to set up assistance systems in intelligent vehicles. It is worth noting too that Li et al. (Li and Wang, 2006) consistently extended $RCC8$ binary networks, and Li (Li, 2007) combined $RCC8$ with another qualitative spatial formalism in order to get a more expressive language.

Renz and Nebel (Nebel, 1995; Renz and Nebel, 1997) showed that the consistency problem of a finite path-consistent network of $RCC8$ constraints is in NP. In order to determine the complexity of a temporal logic based on RCC, such as the one introduced by Wolter and Zakharyaschev (Wolter and Zakharyaschev, 2000), we needed to solve atomic constraints involving an infinite, enumerable number of variables.

Nebel and Bürckert (Nebel and Bürckert, 1995) determined a maximal tractable class of the Interval Algebra introduced by Allen, while Jonsson and Drakengren (Jonsson and Drakengren, 1997), followed by Renz and Nebel (Renz and Nebel, 1997; Renz, 1999), enumerated all the tractable classes of $RCC5$ and $RCC8$. A problem that has never been tackled is the complexity of the consistency of an atomic $RCC8$ constraint network containing an infinite number of variables. In the context of $RCC5$, the above problem was shown to be polynomial in the length of the elements of the network $\mathcal{R}$ (Balbiani, Challita and Condotta, 2003). Due to the specific properties of the $RCC8$ relations EC, TPP and TPP^{-1}; the method developed in (Challita, 2004) appears to be insufficient for answering our question.

Renz (Renz, 1998) already showed that any network of $RCC8$ relations which is consistent has a realization in the n-dimensional Euclidean space $\mathbb{R}^n$ for each $n \geq 1$. Later on, Li (Li, 2006) gave an $O(n^3)$ algorithm for generating a realization of path-consistent networks of $RCC8$ base relations in any $RCC8$ model. More recently, Huang (Huang, 2012) showed that all the maximal tractable fragments of $RCC8$ have patchwork and canonical solutions as long as the networks are algebraically closed. On the other hand, Amaneddine et al. (Amaneddine, Condotta and Sioutis, 2013) proposed an algorithm to derive all the feasible base relations of a qualitative constraint network.

We strongly believe that our work will have some useful and practical applications such as in Geographic Information Systems or a related field, where the number of variables may change over time. For example, consider the situation where we wish to capture (qualitatively) the relative positions of a number of cars that are located in some area. We may use $RCC8$ for this purpose. But since the number of cars can vary over time (cars may enter or leave the area of interest) and there is no upper bound on their number (theoretically we could have an infinite number of variables), we need to design an algorithm that can handle *any* number of variables. The implementation of our algorithm should be inspired by relevant studies (Spall, 1992; Precup, Preitl and Faur, 2003; Attar, Sinha and Wankhade, 2010; Hu and Tan, 2016) in order to be useful for industrial application.

In this paper, we generalize the work done in (Challita, 2012) and propose an incremental algorithm that constructs a realization for any infinite, atomic and path-consistent constraint network of $RCC8$ base relations. Given a partial solution of n variables, we extend it to $n+1$ variables for a total running time of $O((n+1)^3)$. We follow Renz's approach by interpreting the $RCC8$ relations over a topological space. Indeed, and using prime numbers, to each element of such a network, we associate a regular closed subset of the set of real numbers, with its usual topology.

This paper is divided as follows. In Section 2, we recall some basic results concerning $RCC8$. In Section 3, we define the valuation we use to instantiate the variables of any infinite, atomic and path-consistent constraint network of $RCC8$ base relations. We prove in Section 4 that every infinite, atomic and path-consistent network of $RCC8$ base relations is consistent. Before concluding, we give in Section 5 an $O(n^4)$ algorithm for instantiating such networks.

2 The qualitative spatial model $RCC8$

Given a certain number of objects in space, the relations of $RCC8$ enable us to reason about the topological relations that relate them. Denoted by EC, DC, PO, EQ, TPP, $NTPP$, TPP^{-1}, $NTPP^{-1}$, their respective significations for two spatial regions are: "externally connected", "disconnected", "partial overlap", "equal", "tangential proper part", "non-tangential proper part", "tangentially contains" and "strictly contains". They are jointly exhaustive and pairwise disjoint. An example of a spatial representation in the plane of these relations is given in Figure 1.

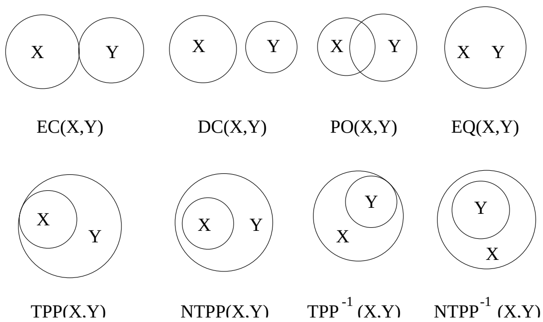


Figure 1: Bi-dimensional representation of the relations of $RCC8$.

Let $\mathcal{R} = (N, C)$ be an $RCC8$ constraint network, where $N \subseteq \mathbb{N}$, and C is a mapping from $N \times N$ to the set of the subsets of $RCC8$ relations. Semantically, $C(x, y)$ contains all the possible relations that are allowed to connect the vertices x and y in $\mathcal{R}$. A model for $\mathcal{R}$ is a structure of the form $\mathcal{M} = (S, v)$, where S is a topological space and v is a valuation that maps the elements of N to non-empty regular closed subsets of S. We say that the model $\mathcal{M}$ satisfies the network $\mathcal{R}$ if for every $i, j \in N$, the relation that holds between $v(i)$ and $v(j)$ in S, and usually denoted by $R_S(v(i), v(j))$, belongs to $C(i, j)$. We next give the definition of the relation $R_S(v(i), v(j))$.

Definition 2.1. Let $\mathcal{R} = (N, C)$ be an $RCC8$ constraint network and $\mathcal{M} = (S, v)$ be a model. $\forall\, i, j \in N$ we have the following:
$EQ_S(v(i), v(j))$ iff $(v(i) = v(j))$,
$TPP_S(v(i), v(j))$ iff $(v(i) \subset v(j) \wedge Fr(i)$ [1] $\cap Fr(j) \neq \emptyset)$,
$TPP_S^{-1}(v(i), v(j))$ iff $TPP_S(v(j), v(i))$,
$NTPP_S(v(i), v(j))$ iff $(v(i) \subset v(j) \wedge Fr(i) \cap Fr(j) = \emptyset)$,
$NTPP_S^{-1}(v(i), v(j))$ iff $NTPP_S(v(j), v(i))$,

[1] $Fr(v) = (\bar{v} \setminus \mathring{v})$ where $\bar{v}$ and $\mathring{v}$ designate the adherence and the interior of v in S.

$DC_S(v(i), v(j))$ iff $(v(i) \cap v(j) = \emptyset)$,
$EC_S(v(i), v(j))$ iff $(\overset{\circ}{v}(i) \cap \overset{\circ}{v}(j) = \emptyset \wedge Fr(v(i)) \cap Fr(v(j)) \neq \emptyset)$,
$PO_S(v(i), v(j))$ iff $(\overset{\circ}{v}(i) \cap \overset{\circ}{v}(j) \neq \emptyset \wedge (\exists\, x \in \overset{\circ}{v}(i), x \notin v(j)) \wedge (\exists\, x \in \overset{\circ}{v}(j), x \notin v(i)))$.

Definition 2.2. A constraint network is consistent if there exists a model that satisfies it. We also say that the valuation v is consistent.

If for all $i, j \in N$ $C(i, j)$ contains exactly one element then the network is said to be atomic. It is path-consistent iff $\forall\, i, j, k \in N$: $C(i, i) = \{EQ\}$, $C(i, j) = C^{-1}(j, i)$ and $C(i, k) \subseteq C(i, j) \circ C(j, k)$. $C^{-1}(i, j)$ contains the inverse of the relations in $C(i, j)$. The composition table of the $RCC8$ relations is given in appendix B. This table first appeared in Cui et al. (Cui, Cohn and Randell, 1993). Renz and Nebel (Renz and Nebel, 1997) used the consistency-based composition of relations to compute it. We next give its definition.

Definition 2.3. Let E be the set of atomic $RCC8$ relations, S a topological space and R_1, $R_2 \in E$.
The relation $R_1 \circ R_2$ is a subset of 2^E which satisfies the following:
$R_1 \circ R_2$ contains all the relations $Q \in E$ such that there exist non-empty closed subsets a, b, c of S such that aR_1b, $b\, R_2\, c$ and aQc hold.

We often wish to provide an answer to the following problem:

Input: a constraint network $\mathcal{R} = (N, C)$.
Output: is there a model $\mathcal{M}$ that satisfies the network $\mathcal{R}$?

The above problem is called $RSAT$. From now on we will consider models of the form $\mathcal{M} = (\mathbb{R}, v)$, where $\mathbb{R}$ is endowed with its usual topology.

3 Infinite $RCC8$ networks

Let $\mathcal{R} = (N, C)$ be a path-consistent network of atomic constraints. For practical reasons, we will denote $v(i)$ and the constraint $C(i, j)$ involving two elements i and j of the network by v_i and C_{ij}, respectively.

The following definitions allow us to simplify the notations of the valuations we will define later on.

Definition 3.1. Given a network $\mathcal{R} = (N, C)$ and an element i of N, let:

$$E_1 = \{j \in N \;:\; C_{ji} \in \{TPP, EQ\}\}, \tag{3.1}$$

$$E_2 = \{(j, j') \in N^2 \;:\; C_{ji} \in \{TPP, EQ\} \wedge C_{jj'} = \{PO\}\}, \tag{3.2}$$

$$E_3 = \{(j, j') \in N^2 \;:\; C_{ji} = \{NTPP\} \wedge C_{jj'} = \{PO\}\}, \tag{3.3}$$

$$E_4 = \{(j, j') \in N^2 \;:\; C_{ji} \in \{TPP, EQ\} \wedge C_{jj'} = \{EC\} \wedge j' > j\}, \tag{3.4}$$

$$E_5 = \{(j, j') \in N^2 \;:\; C_{ij} \in \{TPP, EQ\} \wedge C_{jj'} = \{EC\} \wedge j' < j\} \tag{3.5}$$

$$E_6 = \{j \in N \;:\; C_{ji} = \{NTPP\}\}, \tag{3.6}$$

$$E_7 = \{(j, j') \in N^2 \;:\; C_{ji} = \{NTPP\} \wedge C_{jj'} = \{EC\}\}. \tag{3.7}$$

Later on, the instantiation of each variable x_i of the network will be denoted by v_i. It will be computed in two steps: the first based on $v_i^{'}$ as defined below, and the second based on δ_i as defined in Definition 3.5.

Definition 3.2. $\forall\, j \in \mathbb{N}^*$, let p_j be the j^{th} prime number in $\mathbb{N}$. For example, we have $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.

Definition 3.3. Let $\epsilon \in \mathbb{R},\ 0 < \epsilon < \frac{1}{4}$. $\forall\, i \in N$, keeping the same notations as in definition 3.1, the static valuation is:

$$v_i^{'} = V_{i1} \cup V_{i2} \cup V_{i3} \cup V_{i4}, \tag{3.8}$$

where we have

$$V_{i1} = \left(\bigcup_{j \in E_1} [p_j - \epsilon, p_j + \epsilon] \right) \tag{3.9}$$

$$V_{i2} = \left(\bigcup_{(j,j') \in E_2} [p_j \times p_{j'} - \epsilon, p_j \times p_{j'} + \epsilon] \right) \tag{3.10}$$

$$V_{i3} = \left(\bigcup_{(j,j') \in E_4} [-p_j \times p_{j'} - \epsilon, -p_j \times p_{j'}] \right) \tag{3.11}$$

$$V_{i4} = \left(\bigcup_{(j,j') \in E_5} [-p_j \times p_{j'}, -p_j \times p_{j'} + \epsilon] \right) \tag{3.12}$$

The object of the above definition is to provide a partial instantiation of the networks' variables that involve all the relations of $RCC8$, except the $NTPP$ one.

3.1 Maximal chain

The concept of a maximal chain, as it was introduced in (Challita, 2012), will be used in Definition 3.5.

Definition 3.4. A *chain* is any sequence $T = (\sigma_1, \ldots, \sigma_p) \subseteq N$ containing at least two elements and satisfying: $\forall\, 1 \leq j < p - 1,\ C_{\sigma_j \sigma_{j+1}} = \{NTPP\}$.
A chain T is *maximal with respect to* an element i of the network if $i \in T$ and $\forall\, 1 \leq j \leq p - 1$, there is no k less than i such that:

- $C_{\sigma_j k} = C_{k \sigma_{j+1}} = \{NTPP\}$,
- Or $C_{\sigma_p k} = \{NTPP\}$,
- Or $C_{k \sigma_1} = \{NTPP\}$.

Informally, a chain T is said to be maximal with respect to an element i if it is impossible to "insert" into it any element k of the network that does not appear in T such that $k < i$ and k is in the relation $NTPP$ (resp. $NTPP^{-1}$) with the next (resp. preceding) element of the chain. Notice that there may be several maximal chains with respect to a specific element.

Example 1. Consider the following atomic and path-consistent network where $N = \{1, 2, 3, 4, 5\}$, $C_{21} = C_{31} = C_{32} = C_{34} = C_{51} = \{NTPP\}$, and all the remaining constraints being equal to $\{PO\}$.

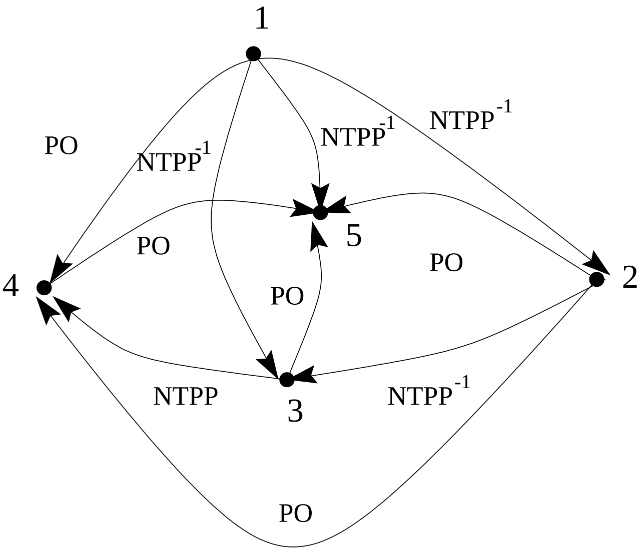


Figure 2: Graphical representation of the network of Example1 .

All the chains of this network are: $T_1 = (3, 2)$, $T_2 = (3, 1)$, $T_3 = (2, 1)$, $T_4 = (3, 2, 1)$, $T_5 = (3, 4)$, $T_6 = (5, 1)$. For instance, T_1 and T_2 are not maximal with respect to the element 3, because despite the fact that $C_{32} = C_{21} = \{NTPP\}$ vertices 1 and 2 do not appear in the chains T_1 and T_2, respectively. On the other hand, the chains T_4, T_5, and T_6 are maximal with respect to $3, 4, 5$, respectively.

3.2 The relation $NTPP$

The aim of the following construction is to solve the problems related to the relation $NTPP$. Suppose that the elements of N are ordered arbitrarily. To each one of them we associate a *radius* $\delta_i \in \mathbb{R}$, computed inductively. In other words, while instantiating the networks' elements, we wish that the following property always holds: $\forall\, i, j \in N, C_{ij} = \{NTPP\} \Rightarrow \delta_i < \delta_j$.

Definition 3.5. Let $\epsilon \in \mathbb{R}$, $0 < \epsilon < \frac{1}{4}$. For all i in N, we inductively compute δ_i as shown below. Let $\delta_0 = \epsilon + \frac{\epsilon}{2}$. Exhaustively, three cases are to be considered:

1. If $\exists\, j < i$ such that $C_{ji} = \{EQ\}$, then $\delta_i = \delta_j$.

2. If $\forall\, j < i,\ C_{ji} \notin \{TPP, TPP^{-1}, NTPP, NTPP^{-1}\}$, then $\delta_i = \delta_0$.

3. If $\exists\, j < i$ such that $C_{ji} \in \{TPP, TPP^{-1}, NTPP, NTPP^{-1}\}$, then we distinguish two cases to compute δ_i.

   (a) $\exists\, j < i$ such that $C_{ji} \in \{NTPP, NTPP^{-1}\}$.
   Consider all the maximal chains $T = (\sigma_1, \ldots, \sigma_p)$ with respect to i, that satisfy: $\forall\ 1 \leq j \leq p,\ \sigma_j \leq i$. Classify them in an arbitrary order and denote them by

$T_l(i) = (\sigma_1^l, \ldots, \sigma_{p_l}^l)$, $(1 \leq l \leq m)$. Three cases are possible: $T_l(i) = (i, \sigma_2^l, \ldots, \sigma_{p_l}^l)$, $T_l(i) = (\sigma_1^l, \ldots, \sigma_{t_l}^l, i, \sigma_{t_l+1}^l, \ldots, \sigma_{p_l}^l)$ or $T_l(i) = (\sigma_1^l, \ldots, \sigma_{p_l-1}^l, i)$. If $\{T_l(i),\ 1 \leq l \leq m\} \neq \emptyset$, then from path-consistency we deduce that one of the previous cases holdÓ.

i. For $T_l(i) = (i, \sigma_2^l, \ldots, \sigma_{p_l}^l)$, let $\delta_i^l = \delta_{\sigma_2^l}$. Define $\delta_i^{'} = min_{(1 \leq l \leq m)}(\delta_i^l - \frac{\epsilon}{4^i})$.

ii. For $T_l(i) = (\sigma_1^l, \ldots, \sigma_{p_l-1}^l, i)$, let $\delta_i^l = \delta_{\sigma_{p_l-1}^l}$. Define $\delta_i^{'} = max_{(1 \leq l \leq m)}(\delta_i^l + \frac{\epsilon}{4^i})$.

iii. For $T_l(i) = (\sigma_1^l, \ldots, \sigma_{t_l}^l, i, \sigma_{t_l+1}^l, \ldots, \sigma_{p_l}^l)$, let $\delta_{i^+}^l = \delta_{\sigma_{t_l+1}^l}$ and $\delta_{i^-}^l = \delta_{\sigma_{t_l}^l}$. Define $\delta_i^+ = min_{(1 \leq l \leq m)}\delta_{i^+}^l$ and $\delta_i^- = max_{(1 \leq l \leq m)}\delta_{i^-}^l$. The value of δ_i^+ follows from some $\gamma_1, ..., \gamma_r$, where $\forall\ (1 \leq l, k \leq r), \delta_{\gamma_l} = \delta_{\gamma_k}$. Let $\sigma = max_{(1 \leq k \leq r)}\gamma_k$. Later on, we will say that δ_i^+ corresponds to σ. In the same way, suppose that δ_i^- corresponds to $\sigma^{'}$. Then we define $\delta_i^{'} = \frac{\delta_i^+ + \delta_i^-}{2}$.

(b) $\exists\, j < i$ such that $C_{ji} \in \{TPP, TPP^{-1}\}$.
Let $S_1(i) = \{j < i\ :\ C_{ij} = \{TPP^{-1}\}\}$ and $S_2(i) = \{j < i\ :\ C_{ij} = \{TPP\}\}$. Three cases are to be considered:

i. For $S_1(i) = \emptyset$ and $S_2(i) \neq \emptyset$, let $\delta_i^{''} = min_{j \in S_2(i)}\delta_j$.

ii. For $S_2(i) = \emptyset$ and $S_1(i) \neq \emptyset$, let $\delta_i^{''} = max_{j \in S_1(i)}\delta_j$.

iii. For $S_1(i) \neq \emptyset$ and $S_2(i) \neq \emptyset$, let $\delta_i^{''} = max_{j \in S_1(i)}\delta_j$ and $\delta_i^{'''} = min_{j \in S_2(i)}\delta_j$.

At this stage, we are able to give the value of δ_i. We distinguish three cases:

(a) If $\forall\, j < i,\ C_{ji} \notin \{TPP, TPP^{-1}\}$ then let $\delta_i = \delta_i^{'}$.

(b) If $\forall\, j < i,\ C_{ji} \notin \{NTPP, NTPP^{-1}\}$, then let $\delta_i = \delta_i^{''}$.

(c) If $\exists\, j, j' < i$ such that $(C_{ji} \in \{TPP, TPP^{-1}\} \wedge C_{ij'} \in \{NTPP, NTPP^{-1}\})$, then we distinguish three subcases:

i. $S_1(i) = \emptyset$. Let $\delta_\mathbf{i} = \mathbf{min}(\delta_\mathbf{i}^{'}, \delta_\mathbf{i}^{''})$.

ii. $S_2(i) = \emptyset$. Let $\delta_\mathbf{i} = \mathbf{max}(\delta_\mathbf{i}^{'}, \delta_\mathbf{i}^{''})$.

iii. $S_1(i) \neq \emptyset$ and $S_2(i) \neq \emptyset$.
For $\delta_i^{''} = \delta_i^{'''}$, let $\delta_\mathbf{i} = \delta_\mathbf{i}^{''}$.
For $\delta_i^{''} \neq \delta_i^{'''}$, we have three possibilities:
- If $\delta_i^{'} > \delta_i^{'''}$, then $\delta_i = \delta_i^{'''}$.
- If $\delta_i^{'} < \delta_i^{''}$, then $\delta_i = \delta_i^{''}$.
- If $\delta_i^{''} \leq \delta_i^{'} \leq \delta_i^{'''}$, then $\delta_i = \delta_i^{'}$.

In this way, to each i in N corresponds a radius δ_i.

4 A consistent instantiation of $RCC8$ networks

The aim of this section is to prove the consistency of any atomic, path-consistent network of $RCC8$ relations.
As we already noted before, our semi-dynamical method for instantiating an element i of N will be done in two steps: (1) first statically solve the problem related to the relation EC, (2) then dynamically solve the one related to the relation $NTPP$, which refers to Definition 3.5.

4.1 Validation of δ_i

We intend to show that the choice of δ_i is convenient. Formally, we must prove the following lemma:

Lemma 1. For all i in N, δ_i is well defined in the following sense:
$\forall\, j \leq i$ and $\forall\, j' < i$, $((C_{jj'} = \{TPP\}) \Rightarrow \delta_j \leq \delta_{j'} \wedge (C_{jj'} = \{NTPP\}) \Rightarrow \delta_j < \delta_{j'})$.

Proof: Denote by P_i the following property:
$\forall\, j \leq i$ and $\forall\, j' < i$, $(C_{jj'} = \{TPP\} \Rightarrow \delta_j \leq \delta_{j'} \wedge C_{jj'} = \{NTPP\} \Rightarrow \delta_j < \delta_{j'})$.
Let us show by induction on i that P_i is true.
Obviously P_1 is true. Suppose that P_{i-1} holds. We must prove that P_i is also true. Three cases are to be considered:
$\mathbf{1^{st}}$**case:** if there exists $j < i$ such that $C_{ji} = \{EQ\}$, then $\delta_i = \delta_j$. Due to the induction hypothesis and to path-consistency, the property P_i is satisfied.
$\mathbf{2^{nd}}$ **case:** for all $j < i$, $C_{ji} \notin \{TPP, TPP^{-1}, NTPP, NTPP^{-1}\}$. By definition we have $\delta_i = \delta_0$.
$\mathbf{3^{rd}}$ **case:** there exists $j < i$ such that $C_{ji} \in \{TPP, TPP^{-1}, NTPP, NTPP^{-1}\}$. Before dealing with this case, we show the following result:

Fact 1. For all i in N, let $T_l(i) = (\sigma_1^l, \ldots, \sigma_{p_l}^l)$ be a maximal chain with respect to i, where $(1 \leq l \leq m)$ for some integer m. If there exists $j < i$ such that $C_{ji} \in \{NTPP, NTPP^{-1}\}$, then: $\forall\, 1 \leq l \leq m,\ \forall\, j \leq p_l,\ ((C_{\sigma_j i} = \{NTPP\} \Rightarrow \delta_{\sigma_j} < \delta_i^{'}) \wedge (C_{i\sigma_j} = \{NTPP\} \Rightarrow \delta_i^{'} < \delta_{\sigma_j}))$.

Proof: $T_l(i)$ being one of the three expressions given previously, three cases are possible:

- $T_l(i) = (i, \sigma_2^l, \ldots, \sigma_{p_l}^l)$: by definition of $\delta_i^{'}$, $\forall\, 1 \leq l \leq m$, $\delta_i^{'} < \delta_{\sigma_2^l}$ holds.
- $T_l(i) = (\sigma_1^l, \ldots, \sigma_{p_l-1}^l, i)$: by definition of $\delta_i^{'}$, $\forall\, 1 \leq l \leq m$, $\delta_i^{'} > \delta_{\sigma_{p_l-1}^l}$ holds.
- $T_l(i) = (\sigma_1^l, \ldots, \sigma_{t_l}^l, i, \sigma_{t_l+1}^l, \ldots, \sigma_{p_l}^l)$: we next check that $\delta_i^- < \delta_i^+$.
  Suppose that δ_i^+ and δ_i^- correspond to σ and $\sigma^{'}$, respectively. We have $C_{\sigma' i} = C_{i\sigma} = \{NTPP\} \overset{p.c.^{2}}{\Longrightarrow} C_{\sigma'\sigma} = \{NTPP\}$. Due to the induction hypothesis, $\delta_{\sigma'} < \delta_\sigma$. The fact 1 is proven.

This ends the proof of Fact 1.
The third case can be decomposed into three subcases.

1. $\forall\, j < i$, $C_{ji} \notin \{TPP, TPP^{-1}\}$. By definition, we have $\delta_i = \delta_i^{'}$ and the property P_i coincides with Fact 1.
2. $\forall\, j < i$, $C_{ji} \notin \{NTPP, NTPP^{-1}\}$. By definition, we have $\delta_i = \delta_i^{''}$ and we easily check that P_i is true.
3. $\exists\, j < i$ and $\exists\, j' < i$ such that $(C_{ji} \in \{TPP, TPP^{-1}\} \wedge C_{ij'} \in \{NTPP, NTPP^{-1}\})$. We distinguish three cases:
   (a) $S_1(i) = \emptyset$. By definition, $\delta_i = \delta_i^{'}$ or $\delta_i = \delta_i^{''}$.

[2] p.c. is an acronym for path-consistency

i. $\delta_i = \delta_i^{'}$. By definition of δ_i, and referring to the induction hypothesis, for all j in $S_2(i)$, $\delta_i \leq \delta_j$. As $\delta_i^{'} = \delta_i$, we deduce from Fact 1 that P_i is satisfied.

ii. $\delta_i = \delta_i^{''}$. By definition, for all j in $S_2(i)$, we have $\delta_i \leq \delta_j$ and $\delta_i^{''} \leq \delta_i^{'}$. From Fact 1 it is easy to see that for all $j < i$ such that $C_{ij} = \{NTPP\}$ we have $\delta_i < \delta_j$. On the other hand, if there exists $k < i$ such that $C_{ki} = \{NTPP\}$ then $\forall\, j \in S_2(i)$, $\forall\, 1 \leq l \leq m$, $(C_{ij} = \{TPP\} \wedge C_{\sigma^l_{t_l} i} = \{NTPP\}) \stackrel{p.c.}{\Longrightarrow} C_{\sigma^l_{t_l} j} = \{NTPP\}$. From the induction hypothesis we have $\delta_{\sigma^l_{t_l}} < \delta_j$. Thus $\delta_{\sigma^l_{t_l}} < \delta_i$.

(b) $S_2(i) = \emptyset$. We proceed in the same way as we did in the previous case.

(c) $S_1(i) \neq \emptyset$ and $S_2(i) \neq \emptyset$. Suppose that $\delta_i^{''}$ and $\delta_i^{'''}$ correspond to j_0 and $j_0^{'}$, respectively. Two cases are possible:

i. $\delta_i = \delta_i^{''} = \delta_i^{'''}$. By definition of δ_i and due to the induction hypothesis we have: $\forall\, j, j' \leq\ i, C_{jj'} = \{TPP\} \Rightarrow \delta_j\ \leq\ \delta_{j'}$. If there exists $k < i$ such that $C_{ik} \in \{NTPP, NTPP^{-1}\}$ then it suffices to show that for all l such that $1\ \leq\ l \leq\ m$ we have $\delta_{\sigma^l_{t_l}}\ <\ \delta_i\ <\ \delta_{\sigma^l_{t_l+1}}$. We already know that $(C_{\sigma^l_{t_l} i}\ =\ \{NTPP\} \wedge C_{ij_0^{'}} = TPP) \stackrel{p.c.}{\Longrightarrow} C_{\sigma^l_{t_l} j_0^{'}} = \{NTPP\}$, and that $(C_{i\sigma^l_{t_l+1}} = \{NTPP\} \wedge C_{j_0 i} = \{TPP\}) \stackrel{p.c.}{\Longrightarrow} C_{j_0 \sigma^l_{t_l+1}} = \{NTPP\}$. Referring to the induction hypothesis we deduce that $\delta_{\sigma^l_{t_l}} < \delta_{j_0^{'}}$ and $\delta_{j_0} < \delta_{\sigma^l_{t_l+1}}$. Thus $\forall\, 1 \leq l \leq m$, $\delta_{\sigma^l_{t_l}} < \delta_i < \delta_{\sigma^l_{t_l+1}}$.

ii. $\delta_i^{''} \neq \delta_i^{'''}$ (i.e. $\delta_i^{''} < \delta_i^{'''}$). For $\delta_i = \delta_i^{'''}$ or $\delta_i = \delta_i^{''}$, the proof is analogous to the one in 3(c)i. For $\delta_i = \delta_i^{'}$, we proceed in the same way as we did in 3(a)i.

5 A consistent valuation of $RCC8$ networks

At this point, we find ourselves able to give a consistent instantiation of any atomic and path-consistent constraint network of $RCC8$.

Definition 5.1. Recall the notations used in Definition 3.1. For $k \in \{3, 6, 7, 8\}$, let F_k be the set E_k with the supplementary condition $(j, j' < i)$. Formally, $F_k = \{(j, j') \in E_i\ :\ j, j' < i\}$ for $k \in \{3, 7, 8\}$ and $F_6 = \{j \in E_6\ :\ j < i\}$.

Given an atomic and path-consistent constraint network $\mathcal{R} = (N, C)$, the algorithm for finding a realization of $\mathcal{R}$ in the set $\mathbb{R}$ is given in Algorithm 1.

Proposition 1. Every atomic and path-consistent network of $RCC8$ base relations containing countably many variables has a realization in time $O(n^3)$.

Proof: Indeed, let $\mathcal{R} = (N, C)$ be a network satisfying the hypotheses of the above proposition. To an element i of the network we associate the valuation v_i. According to Lemma 1 the following is true: $\forall\, j, j' \in N$, $C_{jj'} = \{NTPP\} \Rightarrow \delta_j < \delta_{j'}$. Therefore the valuation v_i is consistent.

Now we turn our attention to determining the running time of the algorithm given in Table 1. Note that we use Iterative Deepening Search (IDS) to compute $v_i^{'}$ since, theoretically speaking, we should be able to instantiate all the $v_i^{'}$ before instantiating any v_i. Practically this is not possible given the fact that our network may contain an infinite number of variables. The idea of using IDS is to allow us to instantiate infinite networks by *re-instantiating* the variables one step

Step 1. Arbitrarily order the elements of N,

Step 2. For each i in N,

(i) Using Iterative Deepening Search, Compute v'_i as described in Definition 3.3,

(ii) Evaluate δ_i,

(iii) Compute v_i as follow:

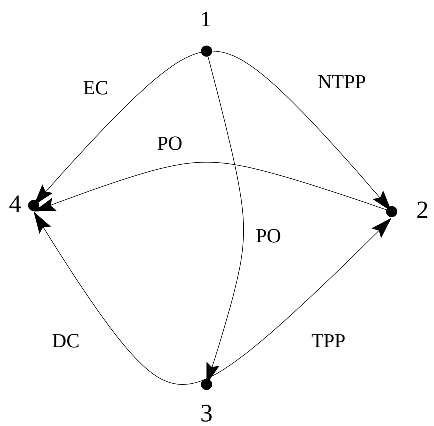
$$v_i = v'_i \cup \left(\bigcup_{j \in F_6} [p_j - \delta_i, p_j + \delta_i]\right)$$
$$\cup \left(\bigcup_{(j,j') \in F_3} [p_j \times p_{j'} - \delta_i, p_j \times p_{j'} + \delta_i]\right)$$
$$\cup \left(\bigcup_{(j,j') \in F_7} [-p_j \times p_{j'} - \delta_i, -p_j \times p_{j'} + \delta_i]\right)$$
$$\cup \left(\bigcup_{(j,j') \in F_8} [-p_j \times p_{j'} - \delta_i, -p_j \times p_{j'} + \delta_i]\right)$$

Table 1: Algorithm for constructing regions v_i

at a time: this is done by increasing the depth of the search space by one at each iteration. It turns out to be a very efficient method in our case and the overhead of these *re-instantiations* of all the network's variables is about 10% (Russel and Norvig, 2009).
Step 1 requires $O(n)$ time, and points (i) and (iii) of Step 2 also require linear time. As for point (ii), it requires time $O(n^2)$. Indeed, and in order to compute δ_i, we need to determine all the maximal chains w.r.t. i (refer to Definition 3.5, case 3(a)). This can be done by a breadth-first search for example, which requires time $O(n^2)$ (see (Cormen, Leiserson, Rivest and Stein, 2001) for more details). Since we need to compute δ_i for every node i of the network $\mathcal{R}$, we deduce that the overall running time of our algorithm is $O(n^3)$.

We next give two examples to describe how our instantiation of atomic and path-consistent constraint networks of $RCC8$ proceeds. In the first two examples we consider finite networks, whereas in the third example we consider an infinite network. For each i in N, the instantiation will proceed in three steps. Firstly determine v'_i, secondly compute δ_i, and finally give the valuation v_i. Notice that while we instantiate a variable i we only consider the constraints $NTPP$ (if they exist) that relate it to the previously instantiated elements of the network.

Example 2. Consider the following network.



According to Definition 3.3, we first have:
$v'_1 = [p_1 - \epsilon, p_1 + \epsilon] \cup [p_1 \times p_3 - \epsilon, p_1 \times p_3 + \epsilon] \cup [-p_1 \times p_4 - \epsilon, -p_1 \times p_4]$.

$v_2^{'} = [p_2 - \epsilon, p_2 + \epsilon] \cup [p_2 \times p_4 - \epsilon, p_2 \times p_4 + \epsilon] \cup [p_3 - \epsilon, p_3 + \epsilon] \cup [p_1 \times p_3 - \epsilon, p_1 \times p_3 + \epsilon]$.
$v_3^{'} = [p_3 - \epsilon, p_3 + \epsilon] \cup [p_1 \times p_3 - \epsilon, p_1 \times p_3 + \epsilon]$.
$v_4^{'} = [p_4 - \epsilon, p_4 + \epsilon] \cup [-p_1 \times p_4, -p_1 \times p_4 + \epsilon] \cup [p_2 \times p_4 - \epsilon, p_2 \times p_4 + \epsilon]$. Secondly, by the definition of δ_i, we have: $\delta_1 = \delta_0 = \epsilon + \frac{\epsilon}{2}$, $\delta_2 = \delta_1 + \frac{\epsilon}{4^2}$, $\delta_3 = \delta_2$ and $\delta_4 = \delta_0$. Finally, referring to Definition 5.1, we conclude that: $v_1 = [2 - \epsilon, 2 + \epsilon] \cup [10 - \epsilon, 10 + \epsilon] \cup [-14 - \epsilon, -14]$, $v_2 = [3 - \epsilon, 3 + \epsilon] \cup [21 - \epsilon, 21 + \epsilon] \cup [5 - \epsilon, 5 + \epsilon] \cup [10 - \epsilon, 10 + \epsilon] \cup [2 - \delta_2, 2 + \delta_2] \cup [10 - \delta_2, 10 + \delta_2] \cup [-14 - \delta_2, -14 + \delta_2]$, $v_3 = [5 - \epsilon, 5 + \epsilon] \cup [10 - \epsilon, 10 + \epsilon]$, $v_4 = [7 - \epsilon, 7 + \epsilon] \cup [-14, -14 + \epsilon] \cup [21 - \epsilon, 21 + \epsilon]$.

A graphical representation of the solution is given in Appendix A.

Example 3. We consider here an infinite network based on an extension of the one given in Example 2. All the given constraints remain the same and we assume that all the new vertices are in the relation PO with all the other nodes of the network. Formally, $\forall\, i \geq 5$, $\forall\, j \geq 1$, $C_{ij} = PO$, where $j \neq i$, since obviously $C_{ii} = EQ$.

According to Definition 3.3, we first have:
$v_1^{'} = [p_1 - \epsilon, p_1 + \epsilon] \cup [p_1 \times p_3 - \epsilon, p_1 \times p_3 + \epsilon] \cup [-p_1 \times p_4 - \epsilon, -p_1 \times p_4] \cup (\bigcup_{j \geq 5} [p_1 \times p_j - \epsilon, p_1 \times p_j + \epsilon])$.
$v_2^{'} = [p_2 - \epsilon, p_2 + \epsilon] \cup [p_2 \times p_4 - \epsilon, p_2 \times p_4 + \epsilon] \cup [p_3 - \epsilon, p_3 + \epsilon] \cup [p_1 \times p_3 - \epsilon, p_1 \times p_3 + \epsilon] \cup (\bigcup_{j \geq 5} [p_2 \times p_j - \epsilon, p_2 \times p_j + \epsilon])$.
$v_3^{'} = [p_3 - \epsilon, p_3 + \epsilon] \cup [p_1 \times p_3 - \epsilon, p_1 \times p_3 + \epsilon] \cup (\bigcup_{j \geq 5} [p_3 \times p_j - \epsilon, p_3 \times p_j + \epsilon])$.
$v_4^{'} = [p_4 - \epsilon, p_4 + \epsilon] \cup [-p_1 \times p_4, -p_1 \times p_4 + \epsilon] \cup [p_2 \times p_4 - \epsilon, p_2 \times p_4 + \epsilon] \cup (\bigcup_{j \geq 5} [p_4 \times p_j - \epsilon, p_4 \times p_j + \epsilon])$.
We notice that the valuations remain the same as in Example 2 except for the last part, which is due to the set E_2 given in Definition 3.1.
Secondly, by the definition of δ_i, we still have: $\delta_1 = \delta_0 = \epsilon + \frac{\epsilon}{2}$, $\delta_2 = \delta_1 + \frac{\epsilon}{4^2}$, $\delta_3 = \delta_2$, $\delta_4 = \delta_0$, since all these values are computed inductively.
Finally, referring to Definition 5.1 and noticing that only the valuation $v(2)$ changes (due to the set E_3 given in Definition 3.1), we conclude that: $v(1) = v_1^{'}$, $v(2) = v_2^{'} \cup (\bigcup_{j \geq 5} [p_1 \times p_j - \epsilon, p_1 \times p_j + \epsilon])$, $v(3) = v_3^{'}$ and $v(4) = v_4^{'}$. As for the remaining valuations, we have:
$\forall\, i \geq 5$, $v(i) = [p_i - \epsilon, p_i + \epsilon] \cup (\bigcup_{j \in \mathbb{N}} [p_i \times p_j - \epsilon, p_i \times p_j + \epsilon])$

6 Conclusion

As we already stated in the introduction, most of the work concerning the complexity and the tractability results of finite networks of $RCC5$ and $RCC8$ relations has been achieved. The problem of determining a consistent instantiation of a path-consistent and atomic constraint network of $RCC5$ relations, containing an infinite number of variables was successfully solved in (Challita, 2004), whereas the same question that concerns the $RCC8$ relations remained unanswered. This problem is of interest to us because we already showed in (Balbiani et al., 2003) that in order to determine the complexity of a spatio-temporal logic introduced by Wolter and Zakharyaschev (Wolter and Zakharyaschev, 2000) that is based on $RCC8$, we need to be able to solve atomic constraint involving an infinite, enumerable number of variables.
In this paper we solved the problem of determining a realization of a path-consistent and atomic constraint network of $RCC8$ base relations, containing an infinite number of variables. Our

proof was done in two steps. We first considered constraint networks containing a finite number of variables, and succeeded in constructing a model that satisfies any such network in time $O(n^2)$. Then we extended our method to infinite networks and gave an $O(n^3)$ algorithm that satisfies any path-consistent and atomic constraint network of $RCC8$ base relations.
Moreover, Renz and Nebel (Renz and Nebel, 1997) showed that every finite and path-consistent constraint network of $RCC8$ whose variables are linked by ORD-Horn relations is consistent. Our next goal is to try to generalize Renz and Nebel's result and provide a consistent instantiation of any atomic ORD-Horn network, possibly containing a countably infinite number of elements.

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A Appendix

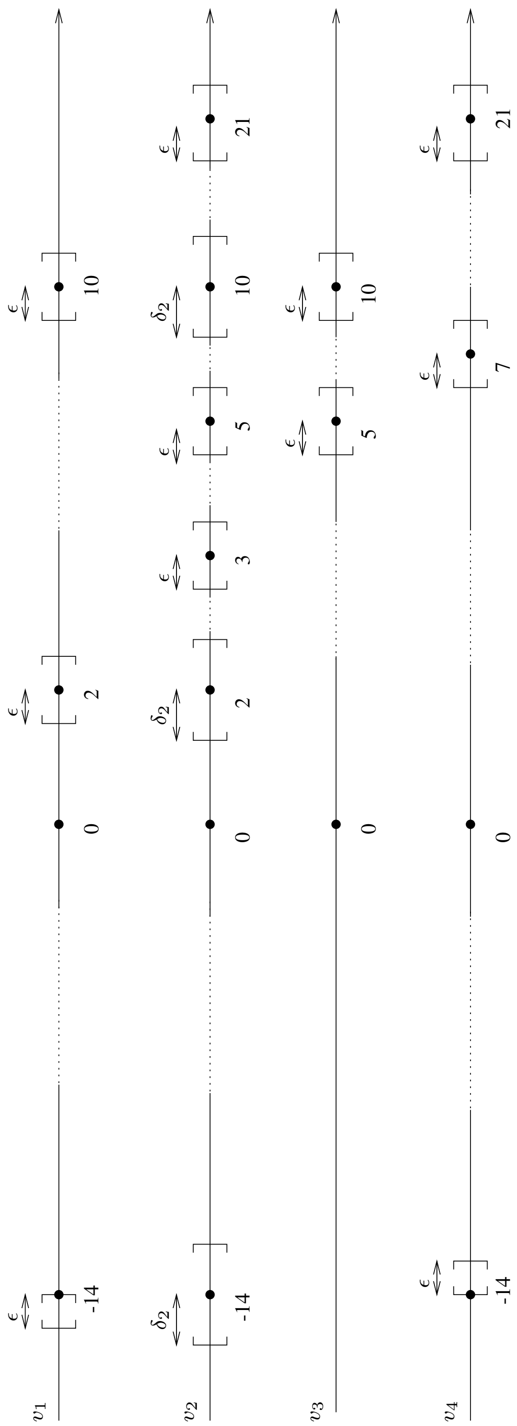

Figure 3: Graphical solution of Example 2, given in Section 4

B Appendix

yRz / xRy	EQ	DC	EC	PO	TPP	NTPP	TPP^{-1}	NTPP^{-1}
EQ	EQ	DC	EC	PO	TPP	NTPP	TPP^{-1}	NTPP^{-1}
DC	DC	*	DC, EC PO, TPP NTPP	DC, EC PO, TPP NTPP	DC, EC PO, TPP NTPP	DC, EC PO, TPP NTPP	DC	DC
EC	EC	DC, EC PO, TPP^{-1} NTPP^{-1}	DC, EC PO, TPP EQ, TPP^{-1}	DC, EC PO, TPP NTPP	PO, TPP EC, NTPP	PO, TPP NTPP	DC, EC	DC
PO	PO	DC, EC PO, TPP^{-1} NTPP^{-1}	DC, EC PO, TPP^{-1} NTPP^{-1}	*	PO, TPP NTPP	PO, TPP NTPP	DC, EC PO, TPP^{-1} NTPP^{-1}	DC, EC PO, TPP^{-1} NTPP^{-1}
TPP	TPP	DC	DC, EC	DC, EC PO, TPP NTPP	NTPP TPP	NTPP	DC, EC PO, TPP EQ, TPP^{-1}	DC, EC PO, TPP^{-1} NTPP^{-1}
NTPP	NTPP	DC	DC	DC, EC PO, TPP NTPP	NTPP	NTPP	DC, EC PO, TPP NTPP	*
TPP^{-1}	TPP^{-1}	DC, EC PO, TPP^{-1} NTPP^{-1}	PO, TPP^{-1} NTPP^{-1} EC	PO, TPP^{-1} NTPP^{-1}	EQ, TPP^{-1} PO, TPP	PO, TPP NTPP	NTPP^{-1} TPP^{-1}	NTPP^{-1}
NTPP^{-1}	NTPP^{-1}	DC, EC PO, TPP^{-1} NTPP^{-1}	PO, TPP^{-1} NTPP^{-1}	PO, TPP^{-1} NTPP^{-1}	PO, TPP^{-1} NTPP^{-1}	EQ, TPP^{-1} NTPP^{-1}, PO NTPP, TPP	NTPP^{-1}	NTPP^{-1}

Figure 4: Composition table of the $RCC8$ relations where the symbol * denotes the union of all possible relations.