$\mathcal{H}_\infty$ Stability of Neural Networks Switched at An Arbitrary Time

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ABSTRACT

This article proposes a novel approach to stability analysis of neural networks switched at an arbitrary time. First, a new condition for $\mathcal{H}_\infty$ stability of switched neural networks is proposed. Second, a new $\mathcal{H}_\infty$ stability condition in the form of linear matrix inequality (LMI) for these neural networks is proposed. These conditions ensure to reduce the $\mathcal{H}_\infty$ norm from the external input to the state vector within a disturbance attenuation level. Without the external input, the proposed conditions also guarantee asymptotic stability.

Keywords: $\mathcal{H}_\infty$ stability, switched neural networks, linear matrix inequality (LMI).

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1 Introduction

As a special class of hybrid systems, switched systems are dynamical systems that include several subsystems and logical rules that orchestrate switching between these subsystems at each instant of time. Switched systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models in manufacturing, communication networks, automotive engine control, and chemical processes. Switched systems have been extensively investigated, and considerable efforts have been focused on the analysis and control of switched systems (Lee, Kim and Lim, 2000; Daafouz, Riedinger and Iung, 2002). Recently, by integrating the theory of switched systems with neural networks, switched neural networks were introduced to describe several complex nonlinear systems efficiently (Huang, Qu and Li, 2005; Yuan, Cao and Li, 2006; Li and Cao, 2007; Lou and Cui, 2007). Some stability conditions for switched neural networks were proposed in (Huang et al., 2005; Yuan et al., 2006; Li and Cao, 2007; Lou and Cui, 2007).

The $\mathcal{H}_\infty$ approach is a powerful technique to analyze the robustness for dynamical systems under conditions of uncertainty, parameter change, and disturbance (Stoorvogel, 1992). Analysis and synthesis in the $\mathcal{H}_\infty$ framework have good advantages such as effective disturbance attenuation, less sensitivity to uncertainties, and many practical applications. This paper gives an answer to the question of whether an $\mathcal{H}_\infty$ stability condition for switched neural networks can be obtained. To the best of our knowledge, the $\mathcal{H}_\infty$ analysis of switched neural networks has not been reported in the literature thus far.

In this paper, we propose new $\mathcal{H}_\infty$ stability conditions for neural networks switched at an arbitrary time. The presented conditions are a new contribution to the stability analysis of switched neural networks. First, a new matrix norm based $\mathcal{H}_\infty$ stability condition is proposed for switched neural networks. Second, a new $\mathcal{H}_\infty$ stability condition in a linear matrix inequality (LMI) form is presented for these neural networks. Under these conditions, the $\mathcal{H}_\infty$ norm from the external input to the state vector is reduced within the $\mathcal{H}_\infty$ norm bound. This paper is organized as follows. In Section 2, new $\mathcal{H}_\infty$ stability conditions are derived. In Section 3, a numerical example is given, and finally, conclusions are presented in Section 4.

2 New Conditions

Consider the following model of switched neural networks (Huang et al., 2005):

$$\dot{x}(t) = A_\alpha x(t) + W_\alpha \phi(x(t)) + J(t), \tag{2.1}$$

where $x(t) = [x_1(t) \ ... \ x_n(t)]^T \in R^n$ is the state vector, $A = diag\{-a_1, \ldots, -a_n\} \in R^{n\times n}$ $(a_k > 0, k = 1, \ldots, n)$ is the self-feedback matrix, $W \in R^{n\times n}$ is the connection weight matrix, $\phi(x(t)) = [\phi_1(x(t)) \ ... \ \phi_n(x(t))]^T : R^n \to R^n$ is the nonlinear function vector satisfying the global Lipschitz condition with Lipschitz constant $L_\phi > 0$, and $J(t) \in R^n$ is an external input vector, α is a switching signal which takes its values in the finite set $\mathcal{I} = \{1, 2, \ldots, N\}$. The matrices (A_α, W_α) are allowed to take values in the finite set $\{(A_1, W_1), \ldots, (A_N, W_N)\}$ at an arbitrary time. In this paper, we assume that the switching rule α is not known a priori and its instantaneous value is available in real time. Define the indicator function $\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_N(t))^T$, where

$$\xi_i(t) = \begin{cases} 1, & \text{when the switched system is described by the i-th mode}(A_i, W_i), \\ 0, & \text{otherwise,} \end{cases}$$

with $i = 1, \ldots, N$. Therefore, the model of the switched neural networks (2.1) can also be written as

$$\dot{x}(t) = \sum_{i=1}^{N} \xi_i(t) \left[A_i x(t) + W_i \phi(x(t)) + J(t) \right], \tag{2.2}$$

where the relation $\sum_{i=1}^{N} \xi_i(t) = 1$ is satisfied under any switching rules.

Given a prescribed level of noise attenuation $\gamma > 0$, the purpose of this paper is to derive conditions such that the switched neural network (2.2) with $J(t) = 0$ is asymptotically stable ($\lim_{t\to\infty} x(t) = 0$) and

$$\int_0^\infty x^T(t)x(t)dt < \gamma^2 \int_0^\infty J^T(t)J(t)dt, \tag{2.3}$$

under zero-initial conditions for all nonzero $J(t) \in L_2[0, \infty)$, where $L_2[0, \infty)$ is the space of square integrable vector functions over $[0, \infty)$.

Now a new $\mathcal{H}_\infty$ stability condition for the switched neural network (2.2) is proposed in the following theorem:

Theorem 2.1. *For a given level $\gamma > 0$, the switched neural network (2.2) is $\mathcal{H}_\infty$ stable if*

$$\|W_i\| < \frac{1}{L_\phi}\sqrt{\frac{k_i - 1 - \frac{1}{\gamma^2}\|P\|^2 - \|P\|}{\|P\|}}, \tag{2.4}$$

$$\|P\| < \frac{-\gamma^2 + \sqrt{\gamma^4 - 4\gamma^2(1-k_i)}}{2}, \qquad k_i > 1, \qquad P = P^T > 0, \tag{2.5}$$

where P satisfies the Lyapunov inequality $A_i^T P + PA_i < -k_i I$ for $i = 1, ..., N$.

Proof. Consider the function $V(t) = x^T(t)Px(t)$. Then, the time derivative of $V(t)$ satisfies

$$\dot{V}(t) < \sum_{i=1}^{N} \xi_i(t)\Big\{ - k_i x^T(t)x(t) + 2x^T(t)PW_i\phi(x(t)) + 2x^T(t)PJ(t)\Big\}. \tag{2.6}$$

By Young's inequality (Arnold, 1989), we have

$$\begin{aligned} 2x^T(t)PW_i\phi(x(t)) &\le x^T(t)Px(t) + (PW_i\phi(x(t)))^T P^{-1}(PW_i\phi(x(t))) \\ &\le \|P\|\|x(t)\|^2 + \|P\|\|W_i\|^2\|\phi(x(t))\|^2 \\ &\le \|P\|\|x(t)\|^2 + L_\phi^2\|P\|\|W_i\|^2\|x(t)\|^2 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} 2x^T(t)PJ(t) &\le \frac{1}{\gamma^2}x^T(t)PP^T x(t) + \gamma^2 J^T(t)J(t) \\ &\le \frac{1}{\gamma^2}\|P\|^2\|x(t)\|^2 + \gamma^2\|J(t)\|^2. \end{aligned} \tag{2.8}$$

If we substitute (2.7) and (2.8) into (2.6), we obtain

$$\begin{aligned} \dot{V}(t) &< \sum_{i=1}^{N} \xi_i(t)\Big\{ - \Big(k_i - \frac{1}{\gamma^2}\|P\|^2 - \|P\| - L_\phi^2\|P\|\|W_i\|^2\Big)\|x(t)\|^2 + \gamma^2\|J(t)\|^2\Big\} \\ &= -\sum_{i=1}^{N} \xi_i(t)\left(k_i - 1 - \frac{1}{\gamma^2}\|P\|^2 - \|P\| - L_\phi^2\|P\|\|W_i\|^2\right)\|x(t)\|^2 \\ &\quad + \sum_{i=1}^{N} \xi_i(t)\Big\{ - \|x(t)\|^2 + \gamma^2\|J(t)\|^2\Big\}. \end{aligned} \tag{2.9}$$

For $i = 1, ..., N$, if the following condition is satisfied:

$$k_i - 1 - \frac{1}{\gamma^2}\|P\|^2 - \|P\| - L_\phi^2\|P\|\|W_i\|^2 > 0, \tag{2.10}$$

we have

$$\dot{V}(t) < -\|x(t)\|^2 + \gamma^2\|J(t)\|^2. \tag{2.11}$$

Integrating both sides of (2.11) from 0 to ∞ gives

$$V(\infty) - V(0) < -\int_0^\infty x^T(t)x(t)dt + \gamma^2\int_0^\infty J^T(t)J(t)dt. \tag{2.12}$$

Since $V(\infty) \ge 0$ and $V(0) = 0$, we have the relation (2.3). The condition (2.10) is rewritten as

$$\begin{aligned} \|W_i\|^2 &< \frac{1}{L_\phi^2\|P\|}\left(k_i - 1 - \frac{1}{\gamma^2}\|P\|^2 - \|P\|\right), \\ 0 &> \|P\|^2 + \gamma^2\|P\| + \gamma^2(1-k_i). \end{aligned} \tag{2.13}$$

The inequality (2.13) implies the inequality (2.5). This completes the proof. ∎

Corollary 2.2. *When $J(t) = 0$, the condition (2.4)-(2.5) ensures that the switched neural network (2.2) is asymptotically stable.*

Proof. When $J(t) = 0$, from (2.11), we have

$$\begin{aligned}\dot{V}(t) &< -\|x(t)\|^2 \\ &< 0, \qquad \forall x(t) \neq 0.\end{aligned} \tag{2.14}$$

This relation ensures that the switched neural network (2.2) is asymptotically stable from Lyapunov stability theory (Precup, Tomescu, Petriu and Dragomir, 2011). This completes the proof. ∎

In the next theorem, a new LMI based $\mathcal{H}_\infty$ stability condition for the switched neural network (2.2) is proposed. The condition in the form of LMI can be facilitated readily via standard numerical algorithms (Boyd, Ghaoui, Feron and Balakrishinan, 1994; Gahinet, Nemirovski, Laub and Chilali, 1995). Hence, this condition is computationally attractive.

Theorem 2.3. *For a given level $\gamma > 0$, the switched neural network (2.2) is $\mathcal{H}_\infty$ stable if there exist a positive symmetric matrix P and a positive scalar ϵ such that*

$$\begin{bmatrix} A_i^T P + PA_i + (\epsilon L_\phi^2 + 1)I & PW_i & P \\ W_i^T P & -\epsilon I & 0 \\ P & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{2.15}$$

for $i = 1, ..., N$.

Proof. Consider the function $V(t) = x^T(t)Px(t)$. By Young's inequality (Arnold, 1989), for any positive scalar ϵ, the following relation is satisfied:

$$\epsilon[L_\phi^2 x^T(t)x(t) - \phi^T(x(t))\phi(x(t))] \geq 0. \tag{2.16}$$

By using (2.16), the time derivative of $V(t)$ is

$$\begin{aligned}\dot{V}(t) \leq& \sum_{i=1}^{N} \xi_i(t) \Big\{ x^T(t)[A_i^T P + PA_i]x(t) + 2x^T(t)PW_i\phi(x(t)) + 2x^T(t)PJ(t) \\ &+ \epsilon[L_\phi^2 x^T(t)x(t) - \phi^T(x(t))\phi(x(t))] \Big\} \\ =& \sum_{i=1}^{N} \xi_i(t) \begin{bmatrix} x(t) \\ \phi(x(t)) \\ J(t) \end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i + (\epsilon L_\phi^2 + 1)I & PW_i & P \\ W_i^T P & -\epsilon I & 0 \\ P & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ \phi(x(t)) \\ J(t) \end{bmatrix} \\ &- x^T(t)x(t) + \gamma^2 J^T(t)J(t).\end{aligned} \tag{2.17}$$

If the LMI (2.15) is satisfied, we have

$$\dot{V}(t) < -x^T(t)x(t) + \gamma^2 J^T(t)J(t). \tag{2.18}$$

Integrating both sides of (2.18) from 0 to ∞ gives

$$V(\infty) - V(0) < -\int_0^\infty x^T(t)x(t)dt + \gamma^2 \int_0^\infty J^T(t)J(t)dt. \tag{2.19}$$

Since $V(\infty) \geq 0$ and $V(0) = 0$, we have the relation (2.3). This completes the proof. ∎

Corollary 2.4. *When $J(t) = 0$, the LMI condition (2.15) ensures that the switched neural network (2.2) is asymptotically stable.*

Proof. When $J(t) = 0$, from (2.18), we have

$$\begin{aligned} \dot{V}(t) &< -x^T(t)x(t) \\ &< 0, \qquad \forall x(t) \neq 0. \end{aligned} \tag{2.20}$$

This inequality ensures that the switched neural network (2.2) is asymptotically stable from Lyapunov stability theory (Precup et al., 2011). This completes the proof. ∎

Remark 2.1. Some stability conditions for switched neural networks were proposed in (Huang et al., 2005; Yuan et al., 2006; Li and Cao, 2007; Lou and Cui, 2007). Despite these advances in stability analysis for switched neural networks, most research results were restricted to switched neural networks without external disturbance. However, in this paper, we apply the $\mathcal{H}_\infty$ approach to derive new stability conditions for switched neural networks with external disturbance

3 Numerical Example

Consider the following switched neural network:

$$\dot{x}(t) = \sum_{i=1}^{2} \xi_i(t) \left[A_i x(t) + W_i \phi(x(t)) + J(t)\right], \tag{3.1}$$

where

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \; J(t) = \begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix}, \; \phi(x(t)) = \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix}, \\ A_1 &= \begin{bmatrix} -3.1 & 0 \\ 0 & -3.5 \end{bmatrix}, \; A_2 = \begin{bmatrix} -3.9 & 0 \\ 0 & -2.8 \end{bmatrix}, \\ W_1 &= \begin{bmatrix} -1 & 0.4 \\ 0 & -0.1 \end{bmatrix}, \; W_2 = \begin{bmatrix} 0.2 & -0.8 \\ 0.4 & 0.5 \end{bmatrix}. \end{aligned} \tag{3.2}$$

By applying Theorem 2.3 via the Matlab LMI Control Toolbox (Gahinet et al., 1995), we have the following feasible solution:

$$P = \begin{bmatrix} 0.5286 & 0.0240 \\ 0.0240 & 0.6075 \end{bmatrix}, \quad \epsilon = 0.5668,$$

with the $\mathcal{H}_\infty$ performance index $\gamma = 0.5$. The switching signal $\alpha \in \{1, 2\}$ is given by

$$\alpha = \begin{cases} 1, & 0 \leq t \leq 2, \\ 2, & \text{otherwise.} \end{cases}$$

When the initial condition is given by $x(0) = [-3.2 \;\; 1.5]^T$ and the external disturbance $J_i(t)$ $(i = 1, 2)$ is given by a Gaussian noise with mean 0 and variance 1, Figure 1 shows the trajectories of state vector $x(t)$. This simulation result confirms the effectiveness of Theorem 2.3 for the $\mathcal{H}_\infty$ stability of the switched Hopfield neural network (3.1).

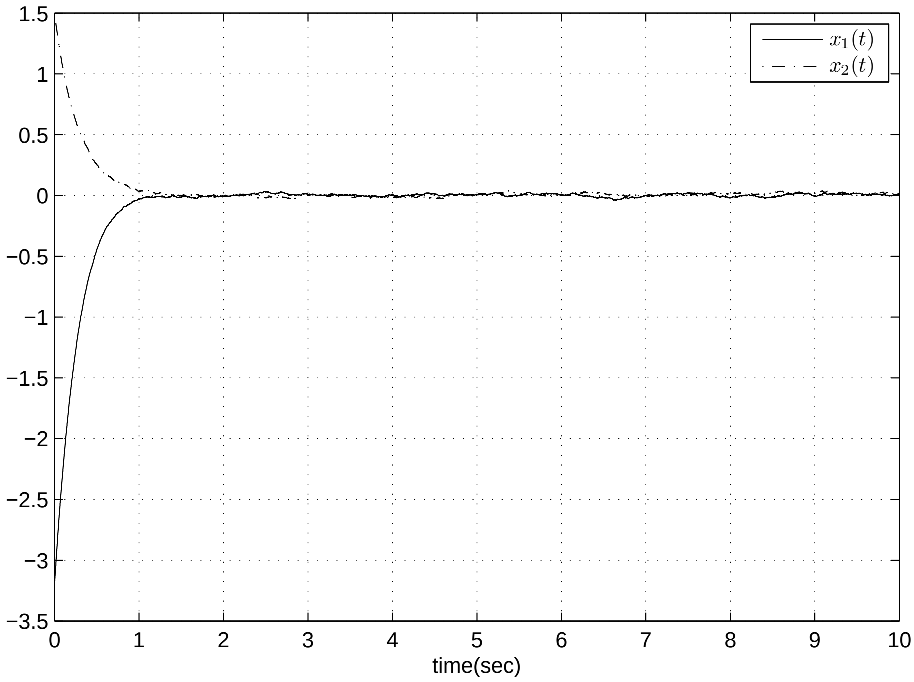

Figure 1: Responses of the state vector $x(t)$

4 Conclusion

In this paper, we propose new matrix norm and LMI based $\mathcal{H}_\infty$ stability conditions for neural networks switched at an arbitrary time. These conditions achieve the $\mathcal{H}_\infty$ performance, with a prespecified attenuation for external input. For external input identically equal to zero, these conditions also guarantee asymptotic stability.

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