This article can be cited as N. Amaneddine and K. Challita, Polynomial Realization of ORD-Horn Constraints, International Journal of Artificial Intelligence, vol. 14, no. 2, pp. 60-69, 2016.

Polynomial Realization of ORD-Horn Constraints

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Abstract

We study in this paper the consistency of a subclass of the Allen's Interval Algebra namely the ORD-Horn networks. More precisely, we consider the Region Connection Calculus RCC5 and RCC8 networks and we prove that these networks have a realization in polynomial time if they satisfy certain conditions. We also prove that subclasses of minimal RCC5 and RCC8 ORD-Horn networks satisfying a specific hypothesis are consistent as well.

Keywords: Qualitative spatial reasoning, CSP, RCC, ORD-Horn.

1998 Computing Subject Classification: F1, F2, F4.

1 Introduction

Qualitative reasoning is an important subfield of Artificial Intelligence and has many practical applications in a wide variety of fields, such as Geographic Information Systems, robot navigation, high level vision and natural languages (Amaneddine and Condotta, 2013), (Amaneddine and Condotta, 2012), (Khmelev and Kochetov, 2015), (Amaneddine, Condotta and Sioutis, 2013), (Wechler, 1992), (Wolter and Zakharyaschev, 2000), (Precup, David, Stînean, Radac and Petriu, 2014), (Precup, David, Petriu, Preitl and Radac, 2013) and (Dong, 2005). Many researchers in the field (e.g. (Ligozat, 1998), (Renz and Nebel, 1997) and (Wolter and Zakharyaschev, 2000)) were inspired by Allen's work on qualitative temporal reasoning (Allen, 1983) and defined other formalisms to reason about objects with respect to space and time. One important qualitative spatial reasoning, namely the Region Connection Calculus was introduced by Randell *et al.* (Randell, Cui and Cohn, 1992). In this qualitative spatial model relationships between spatial regions are defined in terms of the connectivity relation $C(a,b)$. Several subsets of RCC have been studied, amongst them $RCC5$ and $RCC8$ (Bennett, 1994). Jonsson and Drakengren (Jonsson and Drakengren, 1997), followed by Renz and Nebel (Renz, 1999) and (Renz and Nebel, 1997), enumerated all the tractable classes of $RCC5$ and $RCC8$. One important subclass of Allen's Interval Algebra was introduced by Nebel *et al.* (Nebel and Bürckert, 1995): the ORD-Horn subclass. They showed that any finite ORD-Horn network can be instantiated in polynomial time. Moreover, they proved that the ORD-Horn subclass is a maximal tractable subclass of the full algebra (assuming $P \neq NP$). Later on, we proved in (Challita, 2011) that any atomic network $\mathcal{R} = (N, C)$ (finite or infinite) of $RCC5$ (or $RCC8$) constraints that is path-consistent is consistent.

One important problem that was never addressed is the consistency of an infinite ORD-Horn network. In other words, a network that contains an infinite number of variables. We show in this paper that under some specific conditions, such networks have a polynomial-time realization. Moreover, we address the minimality problem of $RCC5$ and $RCC8$ networks and show that they are consistent. Before concluding, we apply our knowledge to solving a problem using a logical agent.

This paper is organized as follows: we describe in Section 2 three important subclasses of RCC, namely $RCC5$, $RCC8$, and the ORD-Horn one. We show in Section 3 that, theoretically speaking, there exists a consistent instantiation of infinite ORD-Horn networks. This result is based on K$\ddot{o}$nig's infinite lemma. In Section 4 we show that any infinite and path-consistent ORD-Horn network that satisfies two hypotheses has a realization in polynomial time. In Section 5, we show that a minimal ORD-Horn network that is path-consistent and infinite has a realization in polynomial time, provided that any strict sub-network of this network contains a finite number of variables which constraints are non-atomic. Before concluding, we give an example in Section 6 of a logical agent that uses Horn clauses for solving a problem in a particular RCC8 ORD-Horn network.

2 RCC and ORD-Horn networks

Two important subsets of RCC are $RCC5$ and $RCC8$. In the former one we can reason about spatial objects using the five basic relations $\{DC, EQ, PP, PPI, PO\}$ without capturing any of their topological properties, whereas in the latter one (i.e. $RCC8$), which can be viewed as a superset of $RCC5$, we have other type of constraints such that *tangential proper part*, and *externally connected*.

A graphical representation of the relations of these two formalisms is given in Figures 1 and 2. Given two regions x and y, the intuitive meaning of the relations of Figure 1 is the following:

- $DC(x, y)$: x and y have an empty intersection.
- $EQ(x, y)$: x and y are equal.
- $PP(x, y)$: x is a subset (i.e. proper part) of y.
- $PPI(x, y)$: y is a subset (i.e. proper part) of x.
- $PO(x, y)$: x and y have a non-empty intersection.

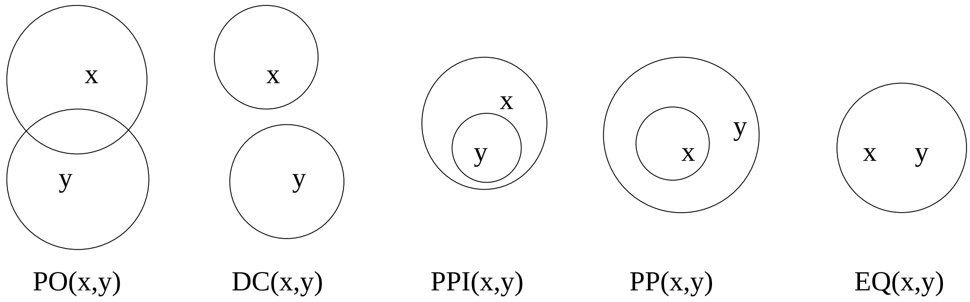

Figure 1: RCC5 basic relations

The intuitive meaning of the relations of Figure 2 is the following:

- $DC(x,y)$: x and y have an empty intersection.
- $EQ(x,y)$: x and y are equal.
- $TPP(x,y)$: x is a subset of y and their boundary is not empty.
- $TPP^{-1}(x,y)$: y is a subset of x and their boundary is not empty.
- $NTPP(x,y)$: x is a strict subset of y and their boundary is not empty.
- $TPP^{-1}(x,y)$: y is a strict subset of x and their boundary is not empty.
- $PO(x,y)$: x and y have a non-empty intersection.
- $EC(x,y)$: the boundary of x and y is not empty.

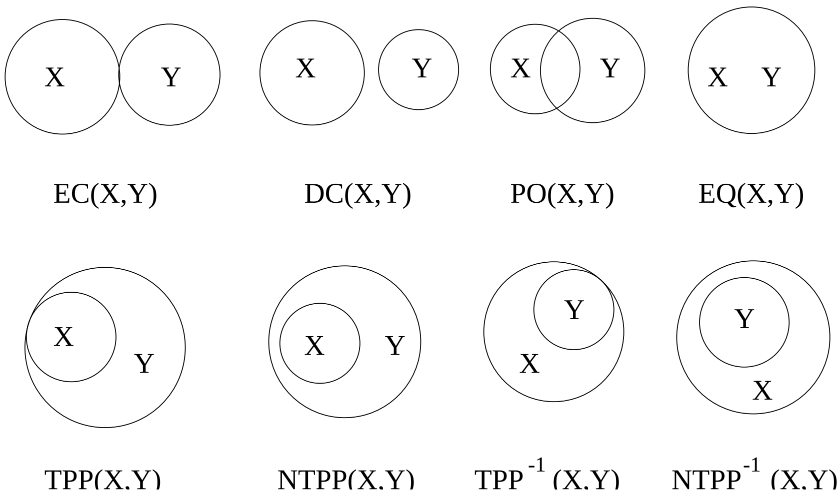

Figure 2: RCC8 basic relations

Nebel *et al.* (Nebel and Bürckert, 1995) introduced a subclass of Allen's Interval Algebra they named ORD-Horn. They proved that path-consistency is sufficient for deciding consistency in polynomial time.

ORD-Horn clauses do not contain negations of atoms of the form $a \leq b$, i.e. they only contain literals of the form: $a = b$, $a \leq b$, and $a \neq b$.
The ORD-Horn clause of an interval formula ϕ is the clause form of ϕ containing only ORD clauses.

3 Consistency of infinite ORD-Horn networks

Let $\mathcal{R} = (N, C)$ be an infinite ORD-Horn network that is path consistent, where the constraints are defined over $RCC5$ (resp. $RCC8$). Based on Renz and Nebel's (Renz and Nebel, 1997) work, we know that any finite and path-consistent ORD-Horn $RCC5$ (resp. $RCC8$) network is consistent. Without loss of generality, we assume that the variables of our network are denoted by $\{1, \ldots, n, \ldots\}$.
After instantiating the first variable, we have at most five different ways to instantiate the second one. In general, we have $O(5^{n^2})$ ways to instantiate the n^{th} variable for the case of $RCC5$ (note that the same is true for $RCC8$, where the number of possibilities is $O(8^{n^2})$).
Indeed, for a $RCC5$ network, if we denote by u_n the maximum number of possible instantiations of the variable n, we have: $u_1 = 1$, $u_2 = 5u_1$, $u_3 = 5^2 u_2, \ldots$, $u_n = 5^{n-1} u_{n-1}$; hence $u_n = 5^{\frac{n(n-1)}{2}}$.
Such a network can be represented by a tree, where each branch corresponds to a possible instantiation of its variables.
For example, the network $N = \{1, 2, 3\}$, where $C_{12} = \{DC, EQ, PP, PPI, PO\}$, $C_{13} = PO$, $C_{32} = \{DC, PP, PO\}$ can be represented by Figure 3, where the vectors at each level of the tree represent possible instantiations of the network's variables.

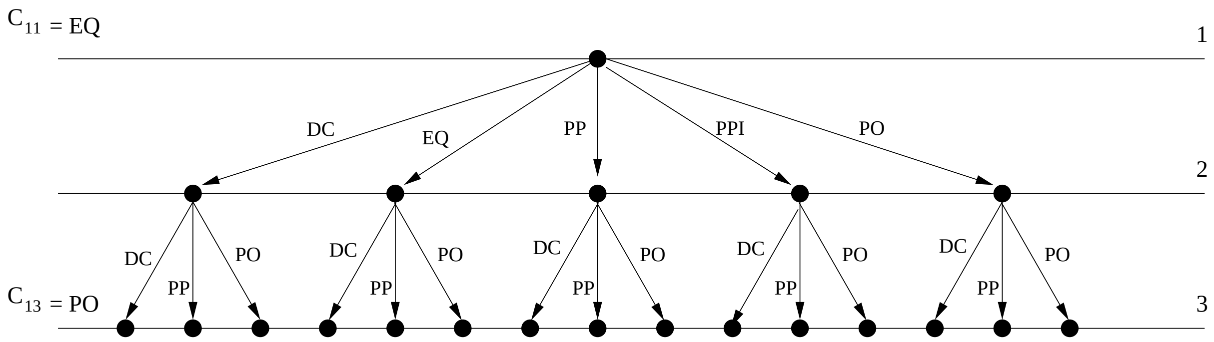

Figure 3: Tree representation of an ORD-Horn network

Recall the following Lemma(Wechler, 1992):

Theorem 1. (König's infinite lemma). Let E be an infinite set that includes one relation denoted by " $\rightarrow$ ". If the sets $E_i \neq \emptyset, i \in I\!N$ form a partition of E such that for all $n \geq 0$ and for all $y \in E_{n+1}$, there exists an element x in E_n that satisfies $x \rightarrow y$, then there exists and infinite chain $a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots$ in E.

Based on this theorem, we can assert that an ORD-Horn path-consistent network $\mathcal{R} = (N, C)$ of $RCC5$ (resp. $RCC8$) constraints is consistent. Indeed, denote by $i \in \mathbb{N}$ the variables of such

a network. For any variable i let the singleton set $E_i = \{i\}$. Furthermore, denote by " $\rightarrow$ " the relation in $RCC5$ (resp. in $RCC8$) such that for all $n \in \mathbb{N}$ and for all $1 \leq i, j \leq n$, the sub-atomic network (that contains only one variable) $\{1, \ldots, n\}$ where $C_{ij} =$ " $\rightarrow$ " is consistent. The choice of the relation " $\rightarrow$ " abovementioned is theoretically possible because based on the work of Renz and Nebel (Renz and Nebel, 1997), we know that any finite and path-consistent ORD-Horn network $\mathcal{R} = (N, C)$ of $RCC5$ (resp. $RCC8$) constraints is consistent.
König's lemma allows us to extend this result to infinite networks. Our next aim is to find an actual instantiation of such networks. In the following section we add some constraints on ORD-Horn networks in order to exhibit a method to instantiate them.

4 Special ORD-Horn networks

In this section we consider ORD-Horn networks with special properties. This is done to allow us to exhibit a method for instantiating such networks, based on the results by Renz and Nebel (Renz and Nebel, 1997) related to finite and path-consistent ORD-Horn network $\mathcal{R} = (N, C)$ of $RCC5$ (resp. $RCC8$).

Definition 1. Let $\mathcal{R} = (N, C)$ be a network of constraints. We define a *path* in $N = \{1, 2, \ldots\}$ any infinite sequence of elements $(p, p+1, \ldots)$ where $p \in N$.

Let $\mathcal{R} = (N, C)$ be an infinite and path-consistent ORD-Horn network of $RCC5$ (resp. $RCC8$) (where $(N \subseteq I\!N^*)$). We assume that such a network satisfies the following two hypotheses:

Hypothesis 1. $\forall\, i \in N$, the set $\{j \in N\ :\ Card(C_{ij}) > 1\}$ is finite.

Hypothesis 2. There is no path $c = (p, p+1, \ldots)$ in N such that $\forall\, j \in c$, Card$(C_{j,j+1}) > 1$.

We are able to find a consistent instantiation for any such network. The idea is to transform the initial network into an atomic one (i.e. with just one constraint between any two of its variables). Recall the following result related to $RCC5$ (and $RCC8$) networks that will be used to prove Proposition 2.

Proposition 1. Any atomic network $\mathcal{R} = (N, C)$ (finite or infinite) of $RCC5$ (or $RCC8$) constraints that is path-consistent is consistent.

This result has been proven in (Challita, 2012).

Proposition 2. Let $\mathcal{R} = (N, C)$ be an ORD-Horn network of $RCC5$ (resp. $RCC8$) that is path-consistent and infinite. If $\mathcal{R}$ satisfies hypotheses 1 and 2 then it is consistent.

Proof. Assume that the elements of N are arbitrarily ordered. We partition this set as follows: for the first element of the network (i.e. for 1 in N), let $X_1 = \{j \in N\ :$ Card$(C_{1j}) > 1\}$. We know that X_1 is finite. If $X_1 = \emptyset$, let $A_1 = \{1\}$, otherwise for all $j \in X_1$, we consider all the paths c_j^α ($\alpha \in E_j$ where the set $E_j \subset I\!N$ is finite) that start at j and that satisfy the condition $\forall\, \alpha \in E_j, \forall\, k \in c_j^\alpha$, Card$(C_{k,k+1}) > 1$. We know that the overall number of such paths is finite.
Let $A_1 = \{1\} \cup \{k \in N\ :\ \exists\, j \in X_1,\ \exists \alpha \in E_j,\ k \in c_j^\alpha\}$.
We consider the smallest element in the set $N \backslash A_1$ (i.e. N minus A_1). Without loss of generality, assume that it is n. As we just done before, let $A_2 = \{n\} \cup \{k \in N : \exists j \in X_n, \exists \alpha \in E_n, k \in c_n^\alpha\}$.
We proceed in the same way to define all the subsequent sets $A_1, \ldots, A_{i-1}$. We next choose a

representative of the set A_i (denoted by n'). For that, we consider the smallest element of the set $N \setminus (\bigcup_{k=1}^{i-1} A_k)$. Note that such an element exists since for all $k \leq i-1$, $Card(A_k)$ is finite. Therefore we have $A_i = \{n'\} \cup \{k \in N \ : \ \exists\, j \in X_{n'},\ \exists \alpha \in E_{n'},\ k \in c^{\alpha}_{n'}\}$.
We can easily check that the sets $\{A_i,\, i \in I\!N^*\}$ form a partition of N. Furthermore, they satisfy the following property: $\forall\, k \in A_j, \forall\, l \in A_i$, $(i \neq j \Rightarrow Card(C_{kl}) = 1)$.
At this point, we are able to suggest a consistent instantiation of N. For all $i \in I\!N^*$, and based on our work (Amaneddine and Condotta, 2013), we know that any A_i network has a consistent instantiation. To conclude our proof, we choose the atomic constraints between the variables of A_i that preserve its consistency. □

5 Minimality problem

Given a network N of variables, the minimal network of N is a network $N' \subseteq N$ where the label between any two variables x, y of N is the minimal relation that is entailed by the network N. In this section we consider minimal networks. We know that there are ORD-Horn networks of $RCC5$ that are path-consistent but not minimal. Figure 4 represents such a network.

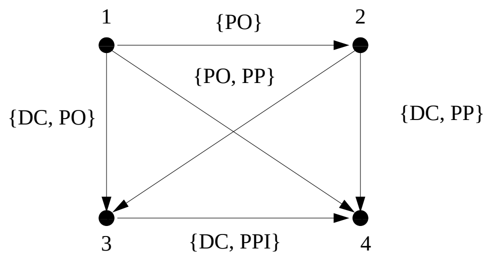

Figure 4: Non-minimal and path-consistent network

Notice that the relation between variables 1 and 4 cannot be reduced to $\{PP\}$. Therefore the polynomial-time path-consistency method is not complete. In order to be able to consistently instantiate a minimal ORD-Horn network we need to enforce some additional hypotheses, as we already did in the previous section.

Hypothesis 3. $\forall\, N' \subsetneq N$, $\{(i,j) \in N' \times N' \ : \ Card(C_{ij}) > 1\}$ is finite.

The above hypothesis states that any strict sub-network of N contains a finite number of variables which constraints are non-atomic. Note that the set $\{(i,j) \in N \times N \ : \ Card(C_{ij}) > 1\}$ may be infinite.

Proposition 3. Let $\mathcal{R} = (N, C)$ be a minimal ORD-Horn network of $RCC5$ (resp. $RCC8$) that is path-consistent and infinite. If this network satisfies hypothesis 3 then it is consistent.

Proof. Assume that the elements of N are arbitrarily ordered and let $N' = N \setminus \{1\}$. According to hypothesis 3, we know that $E = \{(i,j) \in N' \times N' \ : \ Card(C_{ij}) > 1\}$ is finite. Let $F = \{k \in N' \ : \ \exists\, (i,j) \in E, k = i$ or $k = j\}$ and let $\overline{F}$ be the complement of F in N.
Assume that $Card(F \cup \{1\}) = n$. Without loss of generality, we re-order the elements of N in such a way that those who belong to F be placed after the first element. In other words, for

all $i \in F$ and for all $j \in \overline{F} \setminus \{1\}$, $Card(C_{ij}) = 1$. (Refer to Figure 5 to know the type of the constraints of the variables of the network with respect to the first element).

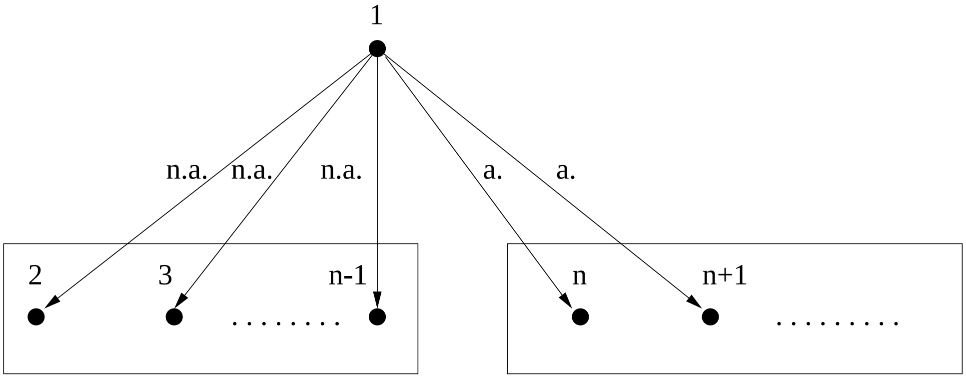

Figure 5: 'a.' and 'n.a.' mean *atomic* and *non-atomic*, respectively.

Since $F \cup \{1\}$ is finite, it has a consistent instantiation. We choose atomic constraints between each couple of variables of $F \cup \{1\}$ in such a way to preserve the path-consistency of the sub-network N'. This is possible since we have: $\forall i, j \in N' \setminus F, \forall k \in F$, $Card(C_{ij}) = Card(C_{ik}) = 1$. To extend this result to N, for all $i \geq n$ we choose C_{1n} in such a way that the sub-network $\{1, \ldots, i\}$ is consistent. This is possible since N is minimal. If the case arises, we apply the triangulation algorithm to the triplets $(1, i, k)$ where $2 \leq \ k \ \leq \ i - 1$ in order to minimize the number of constraints of C_{1i} (we remove those that yield an inconsistent instantiation of the network).

□

6 An application using ORD-Horn networks

In this section we introduce a basic game using a logical agent as defined in (Russell and Norvig, 2009). Recall that a logical agent is able to reason about its surroundings using inference rules in order to update its knowledge base. We start by describing the rules and aims of the game, then explain how to represent and solve it.

A knight's objective is to find and save his princess by avoiding dragons. We assume that he evolves in a two-dimensional space, where the space is discreet (i.e. consists of distinct cells of the same size). Furthermore, the map consists of cells that are of two types: externally connected or disconnected (to be more specific here, any two cells could be in the relation *disconnected* or *externally connected*). The cells are numbered along the x and y axes, and cell i, j is denoted by c_{ij}. An example of such a map is given in figure 6.

For instance, we have $DC(c_{12}, c_{22})$ and $EC(c_{23}, c_{24})$ (i.e. an empty cell is captured by the relation DC and adjacent cells are in the relation EC).

The dragons are hidden in some cells. Therefore, for each cell we have two cases: it contains a dragon or it is empty. We also assume that the dragons are static and do not leave their respective cells. If a cell contains a dragon and the knight is an adjacent cell, then he can sense a dragon's roar. When fighting a dragon the knight has 50% of chances of winning/losing; so it

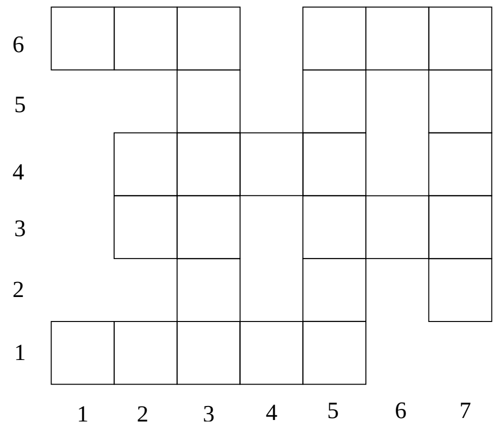

Figure 6: Example of two-dimensional map

is better for him to avoid the cells that contain dragons. At this stage we can clearly define the aim of the knight: find and save the princess by avoiding the dragons.
For example, consider the map in figure 7 where the Knight, the Princess, and the dragons are represented by the letters K, P and D, respectively.

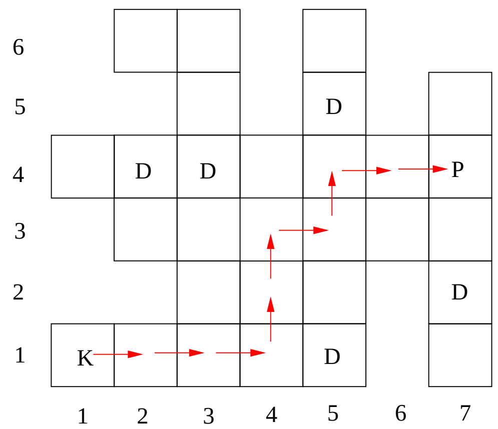

Figure 7: One possible instance of the game.

Note that some instances may not have a solution, depending on the locations of the dragons on the map.
We next explain how to represent and play the game.
It is easy to see that the map can be represented by a an RCC8 network of ORD-Horn constraints where the only two types of relations are EC or DC. The knight starts in the lower left side of the map. It is represented by a logical agent that uses inference rules (e.g. modus ponens[1]) to update its knowledge base in order to make the right decision. We next see how the Knight uses Horn clauses[2] to determine if a cell contains a dragon.
To simplify the notation, we denote by K_{ij} the fact that the Knight is in cell c_{ij} (the same applies for P_{ij} and D_{kl}, to express the fact that the Princess is in cell c_{ij} and a dragon is in cell c_{kl}).

[1] If p is true, and we have $p \Rightarrow q$, then q is also true
[2] A Horn clause is a logical expression of the form $p_1 \wedge \cdots \wedge p_n \Rightarrow q$

Initially, assume that the knight follows the path: $c_{11} \rightarrow c_{12} \rightarrow c_{13} \rightarrow c_{14}$. Each cell is safe and when he reaches the last one he hears the roar of a dragon. He concludes that there is a dragon in c_{42} or c_{51}. He then moves back one cell and follows the path: $c_{32} \rightarrow c_{42}$ that are both safe. At this point in time he is sure that a dragon is in cell c_{15}. This assertion can be derived from the following Horn clause: $Roar_{41} \wedge Safe_{31} \wedge Safe_{42} \Rightarrow D_{51}$.
Notice that a logical agent can solve this instance. After analyzing the map, the Knight is able to find the Princess by following the path with red arrows.

7 Conclusion

In this paper we considered infinite ORD-Horn networks and studied their consistency for the $RCC5$ and $RCC8$ subclasses of the Region Connection Calculus. We already know that finite ORD-Horn networks have a realization in polynomial time. We proved that infinite and path-consistent ORD-Horn networks of $RCC5$ (and also $RCC8$) relations have a polynomial-time instantiation, provided that they satisfy two conditions we stated in Section 4. Furthermore, we showed in Section 5 that any path-consistent, infinite, and minimal ORD-Horn network of $RCC5$ (or $RCC8$) relations is consistent and has a realization in polynomial time. In the last section we gave an example where we used ORD-Horn $RCC8$ networks and Horn clauses for solving a game. Our next step is to extend this result to general ORD-Horn networks (i.e. networks that do not satisfy any special condition) and prove they are consistent.

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